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On the geometry of generalized Robertson–Walker spacetimes: curvature and Killing fields ★

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Abstract

The curvature and Killing vector fields of a class of spacetimes generalizing Robertson–Walker ones (without any assumption on the fiber) is widely studied. Such spacetimes admitting non-trivial Killing vector fields are characterized, and in the globally hyperbolic case, explicitly listed. © 1999 Elsevier Science B.V. all rights reserved.

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1. Introduction

A generalized Robertson–Walker (GRW) spacetime is a *warped product* spacetime where the base is an open interval I of $\mathbb{R}$, with its usual metric reversed $(I, -\mathrm{d}t^2)$, the fiber is an m-dimensional (connected) Riemannian manifold (F, g_F) and the warping function is any positive function $f > 0$ on I. That is, the GRW spacetime is the product manifold $M = I \times F$ endowed with the Lorentz metric

$$g^f = -\mathrm{d}t^2 + f^2(t) g_F, \tag{1.1}$$

★ The main results of this paper were announced, partially, in the meeting Encuentros Relativistas Españoles '97, held at the Universitat Illes Balears, Palma de Mallorca (Spain) from 16 September to 19 September (1997).

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where the natural projections π_I and π_F of $I \times F$ onto I and F, respectively, have been omitted. GRW spacetimes include the usual Robertson–Walker (RW) ones, for which we will assume to have complete fibers of constant curvature. GRW spacetimes were explicitly defined in [2], even though the properties of some warped spacetimes including them have been widely studied in General Relativity; we refer to [16] for motivation and a list of references about their properties. In this paper, we study the properties of their curvature and Killing vector fields; as previous references on this topic (in dimension 4), see [7,11]. Other kind of symmetries of the curvature tensor can be found in [10] (see also [4,8]); for some properties of the curvature of submanifolds, see [1] and references therein.

Our main goal in this paper is to find how to compute all their Killing vector fields. This problem can be stated as follows. Given a Killing vector field on the fiber of a GRW spacetime (or, when f is constant, on the base), a Killing vector field on all the spacetime can be induced naturally; let us call trivial these induced vector fields. Then,

when are there non-trivial Killing vector fields, and in this case, how to construct them?

To answer it, an exhaustive study of Killing vector fields is carried out. Such Killing vector fields have been studied locally in [7], for four-dimensional Lorentzian warped products. Nevertheless, our study in the GRW (n-dimensional) case goes further, because we obtain explicitly both the Killing fields and the GRW spacetimes supporting them; moreover, under the minimum global assumption (completeness of the fiber), strong uniqueness results are given. (On the other hand, we also point out a gap in [7], see the remark in Section 2.)

This paper is organized as follows. In Section 2 we study Ricci curvature, and discuss when a GRW spacetime is Einstein or of constant curvature. There are GRW spacetimes of constant curvature for all $c \in \mathbb{R}$, and there are different ways of writting them as GRW spacetimes (with different warping functions or non-isometric fibers). Nevertheless, in Corollary 2.1 we show that the only complete ones have $c \geq 0$; moreover, the different ways of writting each model space of constant curvature $c \geq 0$ as a GRW spacetime are characterized precisely. We finish Section 2 with two simple consequences in dimension 4, Corollaries 2.2 and 2.3, which extend slightly [11, Theorem 4] and others [9].

Section 3 is technical, and we give there some general properties of Lie derivatives and Killing vector fields in warped spacetimes. Our computations are sometimes parallel to [7], even though ours remain more intrinsic and coordinate free. It is remarkable that we do not use special coordinates for dimension three, which can obscure the computations. In Section 4 the above question is answered, Theorem 4.1, Corollary 4.2, and Theorems 4.3, 4.5 and 4.7. When the spacetime is globally hyperbolic (i.e. the fiber is complete, even though the whole spacetime may not), the structure of the GRW admitting non-trivial Killing vector fields is determined very precisely, Corollaries 4.4, 4.6 and 4.8.

The techniques we use can be applied in more general semi-Riemannian manifolds (i.e., with any index, *including the Riemannian case*, where we have not found analogous results in the literature). So, in Section 3 we work with general warped products; for the sake of completeness, in Section 3.2, the studied elemental properties of Killing fields (some of them well known) are proven or sketched from a general point of view. It is worth pointing out that many of the results through all the paper are valid (or easily adaptable) for warped Lorentzian or semi-Riemannian manifolds with base $(I, +\mathrm{d}t^2)$, including all the "A_{I}-spacetimes" in [7]. We introduce a parameter $\eta = \pm 1$ to keep track of this case: when $\eta = +1$ the result is intended for GRW spacetimes, and when $\eta = -1$ for the case with base $(I, +\mathrm{d}t^2)$. Remarkably, the theorems of Section 4 and Corollary 4.2 hold for $\eta = -1$, but not the other corollaries, except in the (positive) definite case, i.e., when g_F is Riemannian (see the comments between Theorem 4.3 and Corollary 4.4).

2. Conditions on curvature

All the objects will be assumed to be smooth, and all the manifolds connected. The causal character of a tangent vector $v \in T_pM,\ p \in M$ is time-like (resp. light-like, causal) if $g(v, v) < 0$ (resp. $g(v, v) = 0,\ g(v, v) \leq 0$); in general, we will follow the notation and conventions in [5,13].

The general expression of the curvature and Ricci tensors of a general warped product in terms of its base, fiber and warping function is well known [13, Chap. 7]. Particularizing for GRW spacetimes, one has that the Ricci tensor Ric satisfies:

$$\begin{aligned} &\mathrm{Ric}(\partial_t, \partial_t) = -mf''/f, \\ &\mathrm{Ric}(\partial_t, X) = 0, \\ &\mathrm{Ric}(X, Y) = \mathrm{Ric_F}(X, Y) + \eta(f \cdot f'' + (m-1)f'^2) \cdot g_F(X, Y), \end{aligned} \tag{2.1}$$

where X and Y are tangent to the fibers, and $\mathrm{Ric_F}$ is the Ricci of the fiber. As we have already pointed out, the parameter η has been introduced to obtain results including all warped semi-Riemannian manifolds with base $(I, +\mathrm{d}t^2)$. So, the semi-Riemannian warped metric can be regarded as $g^f = -\eta\,\mathrm{d}t^2 + f^2 g_F$; when $\eta = +1$ the result is intended for GRW spacetimes (and all semi-Riemannian warped products with base $(I, -\mathrm{d}t^2)$), and when $\eta = -1$, for semi-Riemannian warped products with base $(I, +\mathrm{d}t^2)$.

From these relations, it is not difficult to show that (M, g^f) is Einstein with $\mathrm{Ric} = mc \cdot g^f$, $m > 1$ $(n = m + 1)$ iff (F, g_F) is Einstein with $\mathrm{Ric_F} = (m-1)c_F \cdot g_F$ and the warping function satisfies:

$$\frac{f''}{f} = \eta c, \qquad f \cdot f'' - f'^2 = \eta c_F. \tag{2.2}$$

If c, c_F are allowed to vary freely, a straightforward computation shows that all the solutions of one of the equations (2.2) are equal to all the solutions of the other one. On the other hand, Eqs. (2.2) are also equivalent to the corresponding set of equations [3, Eq. (3)], whose solutions are tabulated in [3, p. 340]. The solutions to (2.2) can be written as shown in Table 1.

The case $m = 1$ can be considered independently. Recall that the necessary and sufficient condition for the GRW spacetime to be Einstein (i.e., of constant curvature) with curvature c is just the first equation (2.2). Note that one can assume $c_F = 0$, but the second equation

Table 1
Warping functions for $m > 1$; $f > 0$ on $I \subseteq \mathbb{R}$; constants: $A, a, b \in \mathbb{R}$, $A, a \neq 0$; $\eta c = f''/f$, $\eta c_F = f \cdot f'' - f'^2$

1	$\eta c = a^2 > 0$	$\eta c_F = A^2 > 0$	$f(t) = (A/a)\cosh(at + b)$
2	$\eta c = a^2 > 0$	$\eta c_F = 0$	$f(t) = \exp(at + b)$
3	$\eta c = a^2 > 0$	$\eta c_F = -A^2 < 0$	$f(t) = (A/a)\sinh(at + b)$
4	$\eta c = 0$	$\eta c_F = 0$	$f(t) = \exp(b)$
5	$\eta c = 0$	$\eta c_F = -A^2 < 0$	$f(t) = At + b$
6	$\eta c = -a^2 < 0$	$\eta c_F = -A^2 < 0$	$f(t) = (A/a)\cos(at + b)$

Table 2
Warping functions for $m = 1$; $f > 0$ on $I \subseteq \mathbb{R}$; constants: $A_1, A_2, B, B^*, a, b \in \mathbb{R}$, $a, A_1^2 + A_2^2, B^2 + b^2, B^* \neq 0$; $\eta c = f''/f$

1	$\eta c = a^2 > 0$	$f(t) = A_1 \exp(at) + A_2 \exp(-at)$
2	$\eta c = 0$	$f(t) = Bt + b$
3	$\eta c = -a^2 < 0$	$f(t) = B^* \cos(at + b)$

(2.2) may not hold (under the assumption of constant curvature c). Then, for $m = 1$ we can consider the simpler table (Table 2).

Remark. *Writing $f = e^\theta$, the second (and then, both) equation* (2.2) *is equivalent to*

$$\theta'' e^{2\theta} = \eta c_F. \tag{2.3}$$

Eq. (2.3) *has also been considered in* [7, *Section* 4.1] *to study Killing vector fields. Even though these vector fields will be studied in the next section, we can anticipate that, comparing the six cases of Table* 1 *with the formulae* (60) *in* [7, *p.* 474], *our sixth case has not been considered there.*

If we are interested in physically realistic GRW spacetimes, a very natural condition to impose is the *time-like convergence condition* (TCC), or $\mathrm{Ric}(V, V) \geq 0$ for all time-like V. This case can be characterized, replacing the equalities in the Ricci flat case ($c = 0$) by suitable inequalities; so, TCC (for $\eta = +1$) is equivalent to

$$f'' \leq 0, \qquad \mathrm{Ric}_F(X, X) \geq m(f \cdot f'' - f'^2) \cdot g_F(X, X),$$

for all X tangent to the fiber.

To see when the GRW spacetime must be of constant curvature, one has to consider just the warping functions in the tables. Taking into account the expression of the curvature tensor [13, 7.42] it is not difficult to check that the GRW has constant curvature c iff the fiber has constant curvature c_F and the warping function is the corresponding one in the tables (note that these results can be extended to the Riemannian case, and compare with [6, Corollary 7.10; 4, Section III D]). In this case, the GRW spacetime is locally isometric to one of the model spaces: (i) a *de Sitter* spacetime of curvature c, $\mathbb{S}_1^{n*}[c]$, or universal covering of the pseudosphere $\mathbb{S}_1^n[c]$ ($\mathbb{S}_1^{n*}[c] = \mathbb{S}_1^n[c]$ for $n > 2$), when $c > 0$, (ii) *Lorentz–Minkowski* spacetime $\mathbb{R}_1^n$ when $c = 0$, and (iii) an *anti-de Sitter* spacetime of curvature

c, $\mathbb{H}_1^{n*}[c]$, or universal covering of the pseudohyperbolic space $\mathbb{H}_1^n[c]$, when $c < 0$. Now, using the results on completeness in [16] (see also [14]), one can characterize easily which of these model spaces can be written globally as GRW spacetimes.

Corollary 2.1.

(1) *The only global decompositions of the pseudosphere $\mathbb{S}_1^n[c]$, $n > 2$ (resp. $\mathbb{S}_1^2[c]$; $\mathbb{S}_1^{2*}[c]$), $c > 0$, as a GRW spacetime is obtained putting as fiber any usual Riemanian m-sphere of curvature $c_F > 0$ (resp. any circle of radius $R > 0$; $\mathbb{R}$) and as warping function $f(t) = \sqrt{c_F/c}\cosh(\sqrt{c}t + b)$, for any $b \in \mathbb{R}$ (resp. $f(t) = A\exp(\sqrt{c}t) + B\exp(-\sqrt{c}t)$, for any $A, B > 0$).*

(2) *The only global decompositions of Lorentz–Minkowski spacetime $\mathbb{R}_1^n$ as a GRW spacetime is obtained as an usual product (a GRW spacetime with the Euclidean space $\mathbb{R}^m$ as fiber and a positive constant as warping function).*

(3) *No complete Einstein–Lorentzian manifold of negative Ricci curvature (in particular, $\mathbb{H}_1^{n*}[c]$, $\mathbb{H}_1^n[c]$) admits a global decomposition as a GRW spacetime.*

Proof. Assume $m > 1$. By Proposition 4.1 (or its Remark 4) in [16], the fiber must be complete and the only complete cases which occurs in Table 1 are the first and the fourth. So, only these two cases must be taken into account for the global decomposition, proving the last assertion (3). For (1) (resp. (2)) note that the case 1 (resp. 4) must be considered, and the fiber must be complete, simply connected and with constant curvature $c_F > 0$ (resp. $c_F = 0$), i.e., an m-sphere (resp. $\mathbb{R}^m$), as required.

When $m = 1$ the restriction $A, B > 0$ in (1) is imposed to obtain (causal) geodesic completeness (towards both the past and the future). Clearly, a proof completely analogous to the case $m > 1$ works for (2), (3) and $\mathbb{S}_1^{2*}[c]$; for the pseudosphere $\mathbb{S}_1^2[c]$, note that it is not simply connected, but the proof can be modified trivially. □

As an other very simple consequence, we can characterize the four-dimensional Einstein case as follows.

Corollary 2.2. *Let (M, g^f) be a GRW spacetime of dimension $n = 4$. If it satisfies the Einstein vacuum equations with cosmological constant $\lambda \in \mathbb{R}$,*

$$\mathrm{Ric} - (\tfrac{1}{4}R + \lambda)g^f = 0 \tag{2.4}$$

(R scalar curvature) over some non-empty connected set U, then the curvature is constant on the open subset $U' = \pi_I(U) \times \pi_F(U) (\supseteq U)$.

Proof. From (2.4) it follows that the GRW spacetime is Einstein at U. Given $z \in U$ an open neighborhood $N_z \subset U$ can be chosen such that $N_z = \pi_I(N_z) \times \pi_F(N_z)$, and N_z preserves the structure of GRW spacetime. Then, as N_z is GRW Einstein, the open subset $\pi_F(N_z) \subset F$ is Einstein too. But all three-dimensional Einstein (semi-)Riemannian manifolds has constant curvature, and so is $\pi_F(N_z)$. Moreover, the restriction of f to $\pi_I(N_z)$ must belong to Table 1.

Varying $z \in U$ and using the connectedness of U, we have that $\pi_F(U)$ $(= \cup_{z\in U}\pi_F(N_z))$ has constant g_F-curvature. Analogously, using the continuity of the derivatives of f, the restriction of f to $\pi_I(N_z)$ must belong to (just) one of the cases of Table 1. Then, the corresponding warped product on U' must be of constant curvature. □

Clearly, the proof of Corollary 2.2 can be extended to include all four-dimensional A_1-spacetimes (compare with [11, Theorem 4]).

The case in which (F, g_F) is Einstein (even though not necessarily the whole GRW spacetime is) can also be characterized as follows. Recall that, by Eqs. (2.1), ∂_t is a eigenvector of the $(1, 1)$ tensor $\overline{\mathrm{Ric}}$ metrically equivalent to Ric; then, $\overline{\mathrm{Ric}}$ is always diagonalizable. As usual, the GRW spacetime will be called a *perfect fluid* if all the eigenvectors orthogonal to ∂_t at any (t, x) has the same eigenvalue $p(t, x)$.

Corollary 2.3. *The fiber of a GRW spacetime of dimension $n \geq 4$ is Einstein if and only if the spacetime is a perfect fluid; in this case $p(t, x) \equiv p(t)$.*

As a consequence, a globally hyperbolic GRW spacetime of dimension 4 is a perfect fluid if and only if it is a RW.

Proof. The first assertion is a straightforward consequence of (2.1) and the fact that, in dimension $m \geq 3$, if $\mathrm{Ric_F} = \lambda g_F$ for some function λ on F, then λ is constant. For the second assertion, recall that if the fiber is Einstein then it has constant curvature, because of its dimension. □

Note that the global hyperbolicity in Corollary 2.3 is used just because in our definition of RW spacetimes the fibers are assumed complete.

3. Killing fields on semi-Riemannian warped products

3.1. Notation and Lie derivatives

First, we will develop the properties of Killing vector fields on general warped product manifolds, and in the next section, we will focus our attention on GRW spacetimes. So, consider two semi-Riemannian manifolds (B, g_B), (F, g_F) with dimensions $m_B, m_F > 0$ resp. (and arbitrary indexes); let $f = \mathrm{e}^\theta$ denote a positive function on B, and consider the warped metric $g^f = g_B + f^2 g_F$. Given a vector field Z on $B \times F$, we will write $Z = Z_B + Z_F$ with $Z_B = (\pi_{B\star}(Z), 0)$, $Z_F = (0, \pi_{F\star}(Z))$, the projections onto the natural foliations (bases $\{B_q \equiv B \times \{q\}\}$, $q \in F$ and fibers $\{F_p \equiv \{p\} \times F\}$, $p \in B$). Any covariant or contravariant tensor field ω on one of the *factors* (base B or fiber F) induces naturally a tensor field on $B \times F$, which either will be denoted by the same symbol ω, or (just if necessary) will be distinguished by putting a bar on it: $\overline{\omega}$. Latin indexes a, b, c (resp. i, j, k) will run in $1, \ldots, m_B$ (resp. $m_B + 1, \ldots, m_B + m_F$) and will be used for objects tangent to B (resp. F), greek indexes α, β will run in $1, \ldots, m_B + m_F$. So, ∂_a and ∂_i will denote any coordinate vector field for B and F, respectively, and ∂_α any of the previous two (considered, if necessary, on $B \times F$). It is assumed to be the sum in repeated indexes. The symbols L^B, L^F and L represent the Lie derivatives on $B(\equiv B_q, q \in F)$, $F(\equiv F_p, p \in B)$ and $B \times F$, respectively. It is not difficult to check that $L_{Z_B} g_F = L_{Z_F} g_B = 0$, and as a consequence, we have the following lemma.

Lemma 3.1. *For any vector field Z on $B \times F$:*

$$L_Z g^f = L_{Z_B} g_B + Z_B(f^2) g_F + f^2 L_{Z_F} g^F. \tag{3.1}$$

On the other hand, one has:

$$\begin{aligned} L_{Z_B} g_B(\partial_\alpha, \partial_\beta) &= L^B_{Z_B} g_B(\partial_a, \partial_b) \quad \text{if } \alpha, \beta = a, b, \\ L_{Z_B} g_B(\partial_\alpha, \partial_\beta) &= g_B(\partial_a, [\partial_i, Z_B]) \quad \text{if } \alpha = a, \beta = i, \\ L_{Z_B} g_B(\partial_\alpha, \partial_\beta) &= 0 \quad \text{if } \alpha, \beta = i, j. \end{aligned} \tag{3.2}$$

(Note that at each $(p, q) \in B \times F$, we write $L^B_{Z_B} g_B \equiv L^B_{Z_{Bq}} g_B$, where Z_{Bq} is the restriction of Z_B to each base B_q.) Writting an expression analog to (3.2) for the fiber, one has the following lemma.

Lemma 3.2. *Let Z_B, Z_F be two vector fields on $B \times F$ such that Z_B (resp. Z_F) lies in the foliation of the bases (resp. fibers). Then*

$$\begin{aligned} L_{Z_B} g_B(\cdot, \cdot) &= L^B_{Z_B} g_B(\cdot, \cdot) + g_B(\cdot, [\pi_{F\star}(\cdot), Z_B]) + g_B([\pi_{F\star}(\cdot), Z_B], \cdot), \\ L_{Z_F} g_F(\cdot, \cdot) &= L^F_{Z_F} g_F(\cdot, \cdot) + g_F(\cdot, [\pi_{B\star}(\cdot), Z_F]) + g_F([\pi_{B\star}(\cdot), Z_F], \cdot). \end{aligned} \tag{3.3}$$

3.2. Elemental properties

We begin considering properties which are valid in the more general context of submanifolds (compare with [6, Lemma 7.11]). So, take a non-degenerate submanifold S of an arbitrary Lorentzian or semi-Riemannian manifold (M, h).

Lemma 3.3. *If K is a Killing vector field on (M, h) which, at all the points of the submanifold S, is tangent to S then the restriction $K|_S$ is a Killing vector field on S.*

This result is well known (see, e.g. [13, p. 259]), and can be easily proven taking into account that the local fluxes of $K|_S$ are the restrictions of the ones for K, and thus, consists of isometries. Then, for warped products, we have the following proposition.

Proposition 3.4. *If a Killing vector field K on $(B \times F, g^f)$ lies in one of the two natural foliations, then its restriction to each leaf of this foliation is a Killing vector field.*

Moreover, if it lies in the foliation of the bases, then $K(f) \equiv 0$.

Proof. The first assertion is straightforward from Lemma 3.3. For the last one, note that, from (3.1),

$$0 = L_K g^f = L_K g_B + K(f^2) \cdot g_F \tag{3.4}$$

and, by using (3.2), the last two terms vanish (it also reproves the first assertion). □

Remarks

(1) *Clearly, if K is a vector field on $B \times F$ tangent to the bases with $K(f) = 0$ (resp. tangent to the fibers) and such that its restriction to each B_q (resp. F_p) is Killing, it may not be Killing on all $B \times F$ (note that $L_{Z_B} g_B(\partial_a, \partial_i)$ in* (3.2) *may not vanish). The necessary and sufficient condition will be given in Corollary* 3.9 *and Proposition* 3.7.

(2) *Lemma* 3.3 *can be extended easily to conformal Killing vector fields. Moreover, if C is a conformal Killing vector field on $(B \times F, g^f)$ tangent to the bases with conformal expansion σ,*

$$L_C g^f = 2\sigma g^f, \tag{3.5}$$

then C restricted to each base B_q is conformal Killing with $C|_{B_q}(\theta) = \sigma$. By Lemma 3.1, *no conformal Killing vector on $(B \times F, g^f)$ can be tangent to the fibers, but the Killing ones.*

If the Killing vector field K is not tangent to the submanifold, we can consider its projection K_S. But this projection is not a Killing vector field in general.

Lemma 3.5.

(1) *If K is a Killing vector field on (M, h) and S is a totally geodesic submanifold, then K_S is a Killing vector field on S.*

(2) *If C is a conformal-Killing vector field on (M, h) and S is totally umbilical, then C_S is a conformal-Killing vector field.*

Proof. For (2), let $\mathcal{H}$ be the mean curvature vector of the submanifold and σ be the conformal expansion of C. If $\{N_c\}$ is a local basis of the orthonormal bundle to S, then $C_S = C - h^{ab}h(N_a, C)N_b$, and its Lie derivative on any coordinate field ∂_k on S satisfies

$$\begin{aligned} L_{C_S} h(\partial_i, \partial_j) &= L_C h(\partial_i, \partial_j) - L_{(h^{ab}h(N_a,C)N_b)} h(\partial_i, \partial_j) \\ &= 2\sigma h(\partial_i, \partial_j) - h^{ab}h(N_a, C)(h(\nabla_{\partial_i} N_b, \partial_j) + h(\nabla_{\partial_j} N_b, \partial_i)) \\ &= 2(\sigma + h(\mathcal{H}, C))h(\partial_i, \partial_j); \end{aligned} \tag{3.6}$$

so, C_S is conformal Killing. For (1), note that, in (3.6), σ vanishes for a Killing vector field, and if S is totally geodesic, $\mathcal{H}$ vanishes too. □

Remark. *From* (3.6), *when K is Killing and S totally umbilical, K_S is Killing iff $h(\mathcal{H}, K)$ vanishes; in particular, it occurs in the extremal case $\mathcal{H} = 0$.*

Now, as a straightforward consequence for warped products (compare with [6, Lemma 7.11; 7, p. 471]):

Proposition 3.6. *If K is a Killing vector field on $(B \times F, g^f)$, then K_B is a Killing vector field on each B_q, and K_F is a conformal-Killing vector field on each F_p.*

Remark. *In this case, neither K_B is Killing for the warped metric nor it satisfies $K_B(f) = 0$, in general.*

The following property is more specific of warped products.

Proposition 3.7. *Let A, V be vector fields on B, F, and $\overline{A}, \overline{V}$ the natural vector fields induced on $(B \times F, g^f)$, respectively Then,*
(1) *$\overline{A}$ is Killing on $B \times F$ if and only if A is Killing on B and $A(f) = 0$.*
(2) *$\overline{V}$ is Killing on $B \times F$ if and only if V is Killing on F.*

Proof. Both implications to the right are yielded by Proposition 3.4. For the converse, use Lemmas 3.1 and 3.2. □

Remark. *For properly (i.e., non-Killing) conformal Killing vector fields, it is not difficult to check:*
(1) *$\overline{A}$ is properly conformal Killing on $B \times F$ iff A is properly conformal Killing on B and $A(\theta) = \sigma$, where σ is the conformal expansion* (3.5).
(2) *If V is any vector field on F (conformal or not) then $\overline{V}$ is never a properly conformal Killing on the warped product (see Remark* 2 *to Proposition* 3.4).

3.3. Canonical expressions for Killing vector fields

Let $K_{\overline{a}}$, $\overline{a} \in \{1, \ldots, \overline{m}_B\}$ be a basis of Killing vector fields of (B, g_B), and $C_{\overline{\imath}}$, $\overline{\imath} \in \{\overline{m}_B + 1, \ldots, \overline{m}_B + \overline{m}_F\}$ be a basis of conformal-Killing vector fields of (F, g_F), with $L^F_{C_{\overline{\imath}}} g_F = 2\sigma_{\overline{\imath}} g_F$. If K is a Killing vector field on $(B \times F, g^f)$, then, by Proposition 3.6, there are functions $\lambda^{\overline{a}} \equiv \lambda^{\overline{a}}(x^k)$ (resp. $\lambda^{\overline{\imath}} \equiv \lambda^{\overline{\imath}}(x^c)$) depending just on the variables $\{x^k\}$ of F (resp. $\{x^c\}$ of B) such that $K_B = \lambda^{\overline{a}}(x^k) K_{\overline{a}}$ (resp. $K_F = \lambda^{\overline{\imath}}(x^c) C_{\overline{\imath}}$), i.e.,

$$K = \lambda^{\overline{a}}(x^k) K_{\overline{a}} + \lambda^{\overline{\imath}}(x^c) C_{\overline{\imath}}. \tag{3.7}$$

Next, we will characterize the Killing fields among all vector fields on $B \times F$ which can be written as in (3.7). First, note that if ϕ, Z and T are, respectivelly a function, a vector field and a 2-covariant tensor field on M, then,

$$L_{\phi Z} T(\cdot, \cdot) = \phi L_Z T(\cdot, \cdot) + \mathrm{d}\phi(\cdot) \otimes T(Z, \cdot) + T(Z, \cdot) \otimes \mathrm{d}\phi(\cdot). \tag{3.8}$$

Note that, by Lemma 3.2, $L_{K_{\overline{a}}} g_B = L^B_{K_{\overline{a}}} g_B (\equiv 0)$. Thus, by using (3.7) and (3.8):

$$L_{K_B} g_B(\cdot, \cdot) = \mathrm{d}\lambda^{\overline{a}}(\cdot) \otimes g_B(K_{\overline{a}}, \cdot) + g_B(K_{\overline{a}}, \cdot) \otimes \mathrm{d}\lambda^{\overline{a}}(\cdot). \tag{3.9}$$

Analogously for K_F,

$$L_{K_F} g_F(\cdot,\cdot) = 2\lambda^{\bar{1}} \sigma_{\bar{1}}\, g_F(\cdot,\cdot) + \mathrm{d}\lambda^{\bar{1}}(\cdot) \otimes g_F(C_{\bar{1}},\cdot) + g_F(C_{\bar{1}},\cdot) \otimes \mathrm{d}\lambda^{\bar{1}}(\cdot). \quad (3.10)$$

Rewriting the 1-forms $g_B(K_{\bar{a}},\cdot) \equiv \hat{K}_{\bar{a}}$, $g_F(C_{\bar{1}},\cdot) \equiv \hat{C}_{\bar{1}}$, we obtain from (3.1), (3.9), and (3.10):

$$\begin{aligned} L_K g^f &= 2f^2(K_B(\theta) + \lambda^{\bar{1}} \sigma_{\bar{1}})g_F \\ &\quad + (\mathrm{d}\lambda^{\bar{a}} \otimes \hat{K}_{\bar{a}} + \hat{K}_{\bar{a}} \otimes \mathrm{d}\lambda^{\bar{a}}) + f^2(\mathrm{d}\lambda^{\bar{1}} \otimes \hat{C}_{\bar{1}} + \hat{C}_{\bar{1}} \otimes \mathrm{d}\lambda^{\bar{1}}). \end{aligned} \quad (3.11)$$

Finally, taking into account the dependences of the functions λ's in (3.7), we obtain the following result.

Proposition 3.8. *Let K be a vector field on $(B \times F, g^f)$ as in* (3.7). *Then, K is a Killing vector field if and only if the following two relations hold:*

$$K_B(\theta) + \lambda^{\bar{1}} \sigma_{\bar{1}} = 0, \qquad \hat{K}_{\bar{a}} \otimes \mathrm{d}\lambda^{\bar{a}} + f^2\, \mathrm{d}\lambda^{\bar{1}} \otimes \hat{C}_{\bar{1}} = 0. \quad (3.12)$$

The following simple consequence of Proposition 3.8 complements Propositions 3.4 and 3.7 (see also the remarks below Propositions 3.4 and 3.6).

Corollary 3.9. *Let K be a Killing vector field on $(B \times F, g^f)$ which lies everywhere in the foliation of the bases (resp. fibers). Then there exists a Killing vector field A on B (resp. V on F) such that $K = \overline{A}$ and $A(f) = 0$ (resp. $K = \overline{V}$).*

Proof. When K lies in the bases ($K = K_B$), $\lambda^{\bar{1}} \equiv 0$, and from the second equation (3.12), the sum $\hat{K}_{\bar{a}} \otimes \mathrm{d}\lambda^{\bar{a}}$ is null. But, it implies $\mathrm{d}\lambda^{\bar{a}} = 0$ because the 1-forms $\hat{K}_{\bar{a}}$ are independent. Thus, $A = \lambda^{\bar{a}} K_{\bar{a}}$, where $\lambda^{\bar{a}} \in \mathbb{R}$. The assertion on $K(f)$ can be proven either from the first equation (3.12) or from Proposition 3.7.

When K lies in the fibers, reasoning as before the $\lambda^{\bar{1}}$ are constants. Then, use the remark (2) below Proposition 3.7. □

Remark. *A more direct proof to Corollary* 3.9 *can be given from Lemma* 3.2. *Note that, for K as in* (3.7)*:*

$$[\pi_{F\star}(\cdot), K_B] = \mathrm{d}\lambda^{\bar{a}}(\cdot) \otimes K_{\bar{a}}, \qquad [\pi_{B\star}(\cdot), K_F] = \mathrm{d}\lambda^{\bar{1}}(\cdot) \otimes C_{\bar{1}}.$$

From these formulae and Lemmas 3.1 *and* 3.2, *it is clear that if $K = K_B$ (resp. $K = K_F$), then the $\lambda^{\bar{a}}$ (resp. $\lambda^{\bar{1}}$) are constant, from which the result follows.*

4. Classification results for GRW

Now, come back to warped products with one-dimensional base. Even though we will consider GRW spacetimes $(I \times F, g^f)$, the parameter η in Section 2 will be used again to keep track of warped spacetimes with base $(I, +\mathrm{d}t^2)$. For GRW spacetimes, one has

$$K_{\bar{a}} = \partial_t, \qquad \hat{K}_{\bar{a}} = -\eta\, \mathrm{d}t \quad (\overline{m}_B = 1).$$

We will call *trivial* to the Killing vector fields on M obtained from Killing vector fields on the base or fiber by using Proposition 3.7 (or linear combinations of such vector fields); next, we will look for the non-trivial ones. Putting $\lambda = \lambda^1$, $d\lambda^{\bar{\imath}} = \lambda^{\bar{\imath}\,\prime}\, dt$, Eqs. (3.12) can be rewritten as

$$\lambda(x^k)\theta'(t) + \lambda^{\bar{\imath}}(t)\sigma_{\bar{\imath}}(x^k) = 0, \qquad \eta\, d\lambda(x^k) = (f^2 \cdot \lambda^{\bar{\imath}\,\prime})(t)\hat{C}_{\bar{\imath}}(x^k). \tag{4.1}$$

As the 1-forms $\hat{C}_{\bar{\imath}}(x^k)$ are independent, the separation of variables in the second equation (4.1) implies

$$(f^2 \cdot \lambda^{\bar{\imath}\,\prime})(t) = a^{\bar{\imath}} \text{ (constant)}, \tag{4.2}$$

and thus, the second equation (4.1) is equivalent to

$$\lambda^{\bar{\imath}}(t) = a^{\bar{\imath}} \int_{t_0}^{t} f^{-2} + b^{\bar{\imath}}, \quad \text{and} \quad a^{\bar{\imath}} C_{\bar{\imath}} = \eta \nabla^F \lambda, \tag{4.3}$$

for some $t_0 \in I$, $a^{\bar{\imath}}, b^{\bar{\imath}} \in \mathbb{R}$ (∇^F is the g_F-gradient). When $\lambda \equiv 0$, Corollary 3.9 applies, and the Killing field is trivial; otherwise, one can derive with respect to t the first equation (4.1) to obtain, at the points where λ is not null,

$$(f^2 \cdot \theta'')(t) = -a^{\bar{\imath}} \frac{\sigma_{\bar{\imath}}}{\lambda}(x^k) \equiv \eta c_F \text{ (constant)}. \tag{4.4}$$

That is, Eq. (2.3) holds, and we have the following theorem.

Theorem 4.1. *If a GRW spacetime (or warped $\eta = \pm 1$ semi-Riemannian manifold) admits a non-trivial Killing vector field then f must be one of the functions in the tables of Section* 2.

In particular, for dimension $n = 2$ the GRW spacetime has constant curvature. So, this dimension can be characterized directly.

Corollary 4.2. *A simply connected two-dimensional GRW spacetime (or warped $\eta = \pm 1$ semi-Riemannian manifold) admits a non-trivial Killing vector field if and only if it has constant curvature; in this case, the dimension of Killing vector fields is* 3.

In what follows, we will assume $n > 2$.

Now, put $C^* \equiv b^{\bar{\imath}} C_{\bar{\imath}}$; by (3.7) and (4.3),

$$K(t, x^k) = \lambda(x^k)\partial_t + \eta \int_{t_0}^{t} f^{-2} \cdot \nabla^F \lambda + C^*. \tag{4.5}$$

On the other hand, consider the g_F-Hessian of λ, $\mathrm{Hess}_F\lambda(V, W) = g_F(\nabla^F_V \nabla^F \lambda, W)$ $(= (1/2)L^F_{\nabla^F\lambda} g_F(V, W))$. The vector field $C \equiv a^{\bar{\imath}} C_{\bar{\imath}}$ is conformal with expansion $\sigma = a^{\bar{\imath}}\sigma_{\bar{\imath}}$, and by the second equality (4.3) and the last one in (4.4):

$$\mathrm{Hess}_F\lambda + c_F \lambda g_F = 0 \tag{4.6}$$

Summing up, we have obtained that any (non-trivial) Killing vector field K can be written as in (4.5), where f belongs to Table 1, λ is an eigenfunction of the g_F-Hessian, and C^* is a g_F-conformal Killing vector field. Next, we will consider the vector fields given by (4.5), and characterize which of them are non-trivial Killing. The following three cases will be considered, depending on the cases of the warping function $f = \mathrm{e}^{2\theta}$ in Table 1.

(A) Cases $c_F \neq 0$. We can assume that λ is not constant; otherwise, λ would be null by (4.6), and the corrresponding Killing vector fields (4.5) would be trivial, by Corollary 3.9. So, $C = \eta \nabla^F \lambda$ is not identically null. Note that from (4.6), C is conformal with expansion $\sigma = -\eta c_F \lambda$, and we can assume then that C is the first of the $C_{\bar{\mathrm{i}}}$'s. So, all the $a^{\bar{\mathrm{i}}}$ are null, but the first one, which is equal to 1, and the second equation (4.1) holds.

For the first equation (4.1), it is straightforward to check that its left member is independent of t (derive it and obtain 0 by using the expression of the conformal expansion σ, and (2.3)). Then, this equation is satisfied iff the left member vanishes identically for $t = t_0$, i.e., iff

$$\theta'(t_0)\lambda(x^k) + \sigma^*(x^k) \equiv 0, \tag{4.7}$$

where σ^* is the conformal expansion of C^*. But then $\eta c_F \sigma^* - \theta'(t_0)\sigma = 0$, and $C^* - \eta(\theta'(t_0)/c_F)C$ is a (trivial) Killing vector field. We can summarize this as follows.

Theorem 4.3. *Consider a GRW spacetime (or warped $\eta = \pm 1$ semi-Riemannian manifold) with warping function f in* Cases 1, 3, 5 or 6 *of Table 1. Then, its Killing vector fields are given by*

$$K(t, x^k) = \lambda(x^k)\partial_t + \left(\frac{\theta'(t_0)}{c_F} + \eta \int_{t_0}^{t} f^{-2} \right) \nabla^F \lambda + T, \tag{4.8}$$

where λ satisfies (4.6) *and T is a trivial Killing vector field (perhaps null); $t_0 \in I$.*

The existence of solutions to (4.6) is a classical problem in Riemannian geometry. When (F, g_F) is complete, there are powerful results characterizing the only possible cases (as a standard reference, see [17]). So, the following consequence can be stated. (To our knowledge there are no such powerful results when (F, g_F) is Lorentzian, and thus, no analogous result hold for $\eta = -1$ spacetimes.)

Corollary 4.4. *Consider a globally hyperbolic GRW spacetime $(I \times F, g^f)$ admitting a non-trivial Killing vector field K.*

(a) *If f lies in Case* 1 *of Table* 1 *then the fiber is a usual round sphere $\mathbb{S}^m[c_F]$, $m \geq 2$, for the corresponding curvature $c_F > 0$. Thus, if the interval I is maximal ($I = \mathbb{R}$) the spacetime is $\mathbb{S}^n_1[c]$, with the curvature $c > 0$ asssociated to f.*

(b) *If f lies in Cases* 3, 5 *or* 6 *then the fiber (F, g_F) splits as a warped product $\mathbb{R} \times F'$, with base $(\mathbb{R}, \mathrm{d}s^2)$, fiber any complete Riemannian manifold $(F', g_{F'})$ and warping function either $f_F(s) = \Lambda \cosh(\sqrt{-c_F} \cdot s)$, or $f_F(s) = \Lambda \exp(\sqrt{-c_F} \cdot s)$, $\forall s \in \mathbb{R}$, for some constant $\Lambda > 0$.*

The proof is a consequence of Theorem 4.3 and [17, Theorem 2].

Remarks.

(1) *For the sphere $\mathbb{S}^m$, the $m+1$ projections on the coordinate axis are $m+1$ solutions of (4.6), which yield a basis for its conformal Killing vector fields modulo Killing vector fields. When $\mathbb{S}^n_1$ is written as a warped product, these $m+1$ solutions yield a basis of its non-trivial Killing vector fields modulo the trivial ones, by using* (4.8).

(2) *In the case* (b) *of Corollary* 4.4, *when $(F', g_{F'})$ is an hyperbolic spacetime $\mathbb{H}^{m-1}[A^2 c_F]$ (resp. $\mathbb{R}^{m-1}$) then $(F, g_F) = \mathbb{H}^m[c_F]$, and the spacetime is $\mathbb{H}^{n*}_1[c]$, for the corresponding $c < 0$. Note that the $m+1$ independent solutions of* (4.6) *for $\mathbb{H}^m[c_F]$ can be obtained in a similar way than that for the sphere, considering $\mathbb{H}^m$ as a hypercuadric of Lorentz–Minkowski spacetime $\mathbb{R}^n_1$. At any case, the solution λ associated to K can be chosen as the natural projection $\mathbb{R} \times F' \to F'$.*

(B) Case 4, $\theta' \equiv 0$. Again we can assume that λ is not constant (otherwise $\lambda\partial_t$ is trivial, and by Corollary 3.9, so is K). By reasoning as above, the second equation (4.1) is always satisfied, and the first equation holds iff (4.7) holds, i.e., iff C^* is Killing.

Theorem 4.5. *Consider a GRW spacetime (or warped $\eta = \pm 1$ semi-Riemannian manifold) with a constant warping function f_0, Case* 4 *of Table* 1. *Then, its Killing vector fields are given by*

$$K(t, x^k) = \lambda(x^k)\partial_t + \eta(t - t_0) f_0^{-2} \nabla^F \lambda + T, \tag{4.9}$$

where λ has null Hessian and T is a trivial Killing vector field, $t_0 \in \mathbb{R}$.

Remark. *The non-zero parallel gradient field $\nabla^F \lambda$ on the Riemannian manifold (F, g_F) yields a local metric splitting $F = \mathbb{R} \times F'$, $g_F = \mathrm{d}s^2 + g_{F'}$ with $\nabla^F \lambda = \partial_s$; if g_F is complete, the splitting is global. For $\eta = -1$ semi-Riemannian manifolds, this result holds just under additional assumptions; e.g., when $g_F(\nabla^F \lambda, \nabla^F \lambda) \neq 0$* [18] *(see also* [15] *for dimension $m = 2$).*

Then, as a consequence, we have the following corollary.

Corollary 4.6. *Consider a globally hyperbolic GRW spacetime with a constant warping function f_0. If it admits a non-trivial Killing vector field K then the fiber splits metrically $F = \mathbb{R} \times F'$, $g_F = \mathrm{d}s^2 + g_{F'}$, and*

$$K = (as + b)\partial_t + (a f_0^{-2} t + a')\partial_s + T', \tag{4.10}$$

where $a, a' \in \mathbb{R}$, $a \neq 0$, and T' is a trivial Killing vector field, obtained from a Killing vector field on F'.

Remark. *If the dimension of the non-trivial Killing vector fields modulo the trivial ones is l, then F splits metrically as a product $F = \mathbb{R}^l \times F'$ and the functions λ in* (4.9) *are the 1-forms on $\mathbb{R}^l$.*

(C) Case 2, $\theta' \equiv \theta'_0 \neq 0$ ($\Rightarrow c_F = 0$). Consider first that λ is a (non-zero) constant. Note that Eq. (4.6) is trivially satisfied, and (4.1) holds iff Eq. (4.7) holds. Then, C^* must be (properly) homothetic. Thus, this case λ constant occurs iff there exist a homothetic vector field H, $L_H g_F = 2h_0 g_F$, $h_0 \in \mathbb{R} - \{0\}$, and

$$K = h_0 \partial_t - \theta'_0 H. \tag{4.11}$$

If λ is not constant then $\nabla^F \lambda$ is a non-zero parallel field by (4.6), and C^* is properly conformal with $\sigma^* = -\lambda\theta'_0$ by (4.7).

Theorem 4.7. *Consider a GRW spacetime (or warped $\eta = \pm 1$ semi-Riemannian manifold) with a warping function f in Case 2 of Table 1, $\theta' \equiv \theta'_0 (\neq 0)$. Then, any non-trivial Killing vector field K is given by* (4.5), *with λ and C^* satisfying one of the following two possibilities:*

(a) *λ is constant, C^* is homothetic, $C^* = H$, $L_H g_F = 2h_0 g_F$, $h_0 \in \mathbb{R} - \{0\}$, and K satisfies* (4.11).

(b) *$\nabla^F \lambda$ is a non-zero parallel field, and C^* is properly conformal with expansion $\sigma^* = -\theta'_0 \cdot \lambda$.*

Homothetic vector fields yields a Lie algebra of dimension at most one bigger than Killing vector fields. If (F, g_F) is a complete Riemannian manifold then either it is flat or all its homothetic vector fields are Killing [12, p. 242]. Thus, when the GRW is globally hyperbolic, the case (a) in Theorem 4.7 can be applied only if the universal covering of (F, g_F) is $\mathbb{R}^m$.

On the other hand, in the case (b) of Theorem 4.7, the function λ induces a (local) metric splitting $\mathbb{R} \times F'$, as pointed out in the remark below Theorem 4.5. Thus, the projection of C^* onto each slice $\{s_0\} \times F'$ yields a homothetic vector field of this slice. If (F, g_F) is complete then $(F', g_{F'})$ is complete too, and the splitting is global; so, as before, the existence of this non-Killing homothetic vector field on $(F', g_{F'})$ implies that it is flat.

Summing up, we obtain the following result.

Corollary 4.8. *Consider a simply connected globally hyperbolic GRW spacetime with a warping function f in Case 2. If it admits a non-trivial Killing vector field then its fiber is $\mathbb{R}^m$, and thus, the spacetime is an open subset of a pseudosphere $\mathbb{S}^n_1[c]$, $c > 0$.*

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